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AND A BODY OF A ROTATION

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ON THE MUTUAL POTENTIAL OF A SPHERE
AND A BODY OF ROTATION

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In the present paper a form of analysis of the mutual potential of two bodies, suitable for certain applications, is found, having an arbitrary spherical distribution of the density of a sphere and of a body of rotation whose density is any integrand of the distance to the axis of rotation and of the distance to any plane perpendicular to that axis. The analysis obtained can be used in various problems of celestial mechanics, for example in the problem of the translational-rotational motion of an artificial earth satellite or cosmic rocket.

We will consider a certain material body bounded by the area formed by the rotation of a horizontal line around an axis lying in the plane of that line. Let the dimensions of the body be determined by its "length" h and the largest of the radii R .

We will take the axis of rotation to be the axis of the applicates of a rectangular system of coordinates whose origin is in the arbitrary point G of a body lying on the axis of rotation. Let (see Fig. 1)

$$\overline{GC} = l, \quad \overline{GC_1} = l_1, \quad l + l_1 = h,$$

and we write the equation for the meridian section of the body in the form

$$\rho = R(\zeta),$$

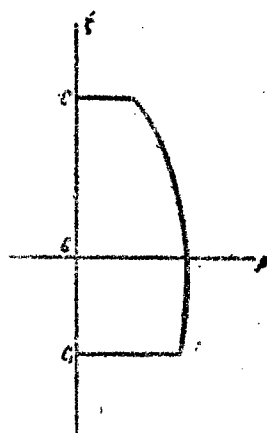


Fig. 1

where $R(\xi)$ is the given integrand of ξ , determined in the interval $(-\ell_1, \ell)$.

Let the density of the body be a given integrand $\delta(\rho', \xi')$ for the distance ρ' to the axis of rotation and the distance ξ' to the basic plane. Let us imagine a circumference passing through an arbitrary point M of a body with its center on the axis of rotation.

Then

$$dm = 2\pi \delta(\rho', \xi') \rho' d\xi' d\rho'$$

can be accepted as an element of the mass of our body. Obviously the total mass m is determined by the formula

$$m = 2\pi \int_{-\ell_1}^{\ell} d\xi' \int_0^{R(\xi')} \delta(\rho', \xi') \rho' d\rho'. \quad (1)$$

Since G is an arbitrary point of the body lying on the axis of rotation, ℓ and ℓ_1 are any numbers whose sum equals the length of the body h . If, in particular, G is the center of inertia of the body, then ℓ and ℓ_1 will be fully determined. In fact, if we take an origin of the coordinates in the center of the lower base C_1 we will have, for example

$$m\ell_1 = 2\pi \int_0^h \xi_1 d\xi_1 \int_0^{R(\xi_1)} \delta_1(\rho', \xi_1) \rho' d\rho', \quad (2)$$

where

$$\xi' = \xi_1 - \ell_1, \quad R_1(\xi_1) = R(\xi_1 - \ell_1), \quad \delta_1(\rho', \xi_1) = \delta(\rho', \xi_1 - \ell_1)$$

and where m , in place of formula (1), can also be determined by the formula

$$m = 2\pi \int_0^h d\xi_1 \int_0^{R_1(\xi_1)} \delta_1(\rho', \xi_1) \rho' d\rho'.$$

If now in formula (2) we go back to the system of coordinates with its origin in the center of inertia of the body G we get the obvious equality:

$$0 = \int_{-\ell_1}^{\ell} \xi' d\xi' \int_0^{R(\xi')} \delta(\rho', \xi') \rho' d\rho'. \quad (3)$$

In the future we will consider point G an arbitrary point lying on the axis of rotation of the body (inside the body).

Now let P be any point of the axis of the applicates which is external as regards the body and at which the mass m_0 is centered.

Or let the point of the axis of the applicates P be the center of a sphere, homogeneous or possessing spherical distribution of densities, whose mass equals m_0 and whose radius is a less than the distance of point P to the point of the body nearest it. Such a sphere, as is known, attracts and is attracted as a material point of mass m_0 located at point P. Then

$$\overline{PM}^2 = \rho'^2 + (\zeta - \zeta')^2$$

and the potential of an infinitely thin round ring with mass dm on point P will be

$$\frac{\int m_0 dm}{PM} = \frac{2\pi f m_0 \delta(\rho', \zeta') \rho' d\zeta' d\rho'}{\sqrt{\rho'^2 + (\zeta - \zeta')^2}}.$$

and the potential of the whole body on the same point P is determined by the formula

$$U(\zeta) = 2\pi f m_0 \int_0^{\zeta} \int_0^{R(\zeta')} \frac{\delta(\rho', \zeta') \rho' d\zeta' d\rho'}{\sqrt{\rho'^2 + (\zeta - \zeta')^2}} \quad (4)$$

Now in accordance with the well-known theorem of classical theory of the point potential [1] we obtain the potential of the body in all the external space (external in relation to some closed surface containing our body within itself), if in an absolutely convergent expansion of the potential $U(\zeta)$ by steps c/ζ we replace ζ^{-k-1} by $X_k(\cos \vartheta) \times r^{-k-1}$. Here C designates the largest of the values $\ell \geq \ell_1$ and R ; $X_k(\cos \vartheta)$ is the k Legendre polynom and ϑ is the angle between the radius-vector r of point P and the positive direction of the axis of the applicates.

To obtain the necessary expansion of the potential $U(\zeta)$ we will examine the reverse distance under the integral sign in the formula (4). Assuming $r'^2 = \rho'^2 + \zeta'^2$, we can write

$$\frac{1}{\sqrt{\rho'^2 + (\zeta - \zeta')^2}} = \frac{1}{\zeta} \left[1 - 2 \frac{\zeta'}{\zeta} \frac{\rho'}{\zeta} + \left(\frac{\rho'}{\zeta} \right)^2 \right]^{-\frac{1}{2}}.$$

Since $\left| \frac{\zeta'}{\zeta} \right| \leq 1$, at $\left| \frac{\rho'}{\zeta} \right| < 1$ we have an absolutely convergent expansion

$$\left[1 - 2 \frac{\zeta'}{\rho'} \cdot \frac{\rho'}{\zeta} + \left(\frac{\rho'}{\zeta} \right)^2 \right]^{-\frac{k}{2}} = \sum_{k=0}^{\infty} X_k \left(\frac{\zeta'}{\rho'} \right) \frac{\rho'^k}{\zeta^k},$$

with the help of which we will present the reverse direction in the form

$$\frac{1}{\sqrt{\rho'^2 + (\zeta - \zeta')^2}} = \sum_{k=0}^{\infty} a_k \frac{c^k}{\zeta^{k+1}} \quad (5)$$

with the dimensionless coefficients of a_k determinable by the formula

$$a_k = \frac{\rho'^k}{c^k} X_k \left(\frac{\zeta'}{\rho'} \right) = \frac{1}{c^k} \sum_{s=0}^{E\left(\frac{k}{2}\right)} A_{ks} (\zeta')^{k-2s} (\rho'^2 + \zeta'^2)^s, \quad (6)$$

where $E\left(\frac{k}{2}\right)$ is an integral part of the number $\frac{k}{2}$ and A_{ks} which are the numerical coefficients:

$$A_{ks} = (-1)^s \frac{(2k-2s)!}{2^k s! (k-s)! (k-2s)!}.$$

We note that sequence (5) is known to be converging absolutely if $|\zeta|$ is many times larger than C .

If we substitute now, in place of the reverse distance, its expansion (5), we get the required expansion of the potential $U(\zeta)$ in the form:

$$U(\zeta) = 2\pi f m_0 \sum_{k=0}^{\infty} \frac{c^k}{\zeta^{k+1}} \int_{l_1}^l \int_0^{R(\zeta')} a_k(\rho', \zeta') \delta(\rho', \zeta') \rho' d\zeta' d\rho'.$$

This expansion converges absolutely if $|\zeta| > c$. In fact, from formula (6) and taking into consideration the properties of the Legendre polynom, we have for total k the inequality $|a_k| \leq 1$, and therefore

$$\left| \int_{l_1}^l \int_0^{R(\zeta')} a_k(\rho', \zeta') \delta(\rho', \zeta') \rho' d\zeta' d\rho' \right| < m,$$

which also proves our assertion.

By applying now the above-mentioned theorem of the theory of potential we get an expansion of the potential of our body on any external point P in the form:

$$U(P) = 2\pi f m_0 \sum_{k=0}^{\infty} \frac{c^k X_k(\cos \theta)}{r^{k+1}} \int_{-1}^1 \int_0^{R(\zeta')} a_k(\rho', \zeta') \delta(\rho', \zeta') \rho' d\zeta' d\rho'.$$

Let us interpolate now instead of ρ' the angle $\chi = \pi - \theta$, that is, the angle between the direction going from point P (which can be in the center of the sphere possessing a spherical distribution of density) to point G, and the direction of the axis of rotation of the body, which is assumed to be the positive direction of the axis of the applicates. Then

$$\cos \theta = -\cos \chi = -\nu, \quad X_k(\cos \theta) = (-1)^k X_k(\nu),$$

and the mutual potential of the sphere and the body of rotation will depend only on r and ν .

We will present the final expression of the potential in the form:

$$U(r, \nu) = \frac{f m_0 m}{r} \sum_{k=0}^{\infty} A_k X_k(\nu) \frac{c^k}{r^k}, \quad (7)$$

where the coefficients are determined by the following general formula:

$$A_k = (-1)^k \frac{2\pi}{m} \int_{-1}^1 \int_0^{R(\zeta')} a_k(\rho', \zeta') \delta(\rho', \zeta') \rho' d\zeta' d\rho'. \quad (8)$$

In agreement with classical theory the sequence (7) converges absolutely at any $|\nu| \leq 1$ and at $r > c$, if, of course, it is a matter of point mass m_0 centered on P. Since we are interested in a case of a sphere with its center at point P, sequence (7) will absolutely converge at $|\nu| \leq 1$ and at $r > c + R_0$, where R_0 is the radius of the sphere.

Using formula (6) for the coefficients of X_k , we write the first coefficients of sequence (7). We have

$$\begin{aligned} A_0 &= 1, \quad A_1 = -\frac{2\pi}{mc} \int_{-1}^1 \int_0^{R(\zeta')} \zeta' \delta(\rho', \zeta') \rho' d\zeta' d\rho', \\ A_2 &= \frac{2\pi}{mc^2} \int_{-1}^1 \int_0^{R(\zeta')} \left(\zeta'^2 - \frac{1}{2} \rho'^2 \right) \delta(\rho', \zeta') \rho' d\zeta' d\rho', \\ A_3 &= -\frac{2\pi}{mc^3} \int_{-1}^1 \int_0^{R(\zeta')} \zeta' \left(\zeta'^2 - \frac{3}{2} \rho'^2 \right) \delta(\rho', \zeta') \rho' d\zeta' d\rho'. \end{aligned}$$

We note here that if we take the center of inertia of the body as point G the equality (3) is fulfilled and it follows that in this case $A_1 = 0$.

Let us now use C to designate the moment of inertia of the body relative to the axis of rotation, and A the moment of inertia relative to any axis passing through the center of inertia G perpendicular to the axis of rotation. Simple calculation gives:

$$C = 2\pi \int_{-l_1}^{l_1} \int_0^{R(\zeta)} \delta(\rho', \zeta) \rho'^3 d\zeta' d\rho',$$

$$A = 2\pi \int_{-l_1}^{l_1} \int_0^{R(\zeta)} \left(\zeta'^2 + \frac{1}{2} \rho'^2 \right) \delta(\rho', \zeta') \rho' d\zeta' d\rho'.$$

from which it follows that

$$A_2 = \frac{A - C}{mc^2}.$$

If, furthermore, the body possesses geometrical and mechanical symmetry relative to the center of inertia G, that is, if

$$l = l_1, \quad R(-\zeta) = R(\zeta), \quad \delta(\rho', -\zeta) = \delta(\rho', \zeta),$$

then, as formulas (6) and (7) show, all the odd coefficients are equal to zero and the even are determined by the formulas:

$$A_{2k} = \frac{4\pi}{mc^{2k}} \sum_{s=0}^k \times$$

$$\times A_{2k,s} \int_0^{l_1} \int_0^{R(\zeta)} \zeta'^{2k-2s} (\rho'^2 + \zeta'^2)^s \delta(\rho', \zeta') \rho' d\zeta' d\rho',$$

and in this case the expansion of the potential is presented in the form:

$$U(r, \nu) = \frac{f m_a m}{r} \sum_{k=0}^{\infty} A_{2k} X_{2k}(\nu) \frac{c^{2k}}{r^{2k}} \quad (9)$$

The constant c in the formulas designates the greater of the two values $l \geq l_1$ and R.

If $l \geq R$ we will for the sake of brevity call the body

"elongated" or "lengthened", and will write the corresponding expansion of the potential, assuming $c = \ell$ in general, in the form:

$$U(r, \nu) = \frac{fm_0m}{r} \sum_{k=0}^{\infty} A_k X_k(\nu) \frac{\ell^k}{r^k}.$$

and for a body possessing symmetry relative to the center of inertia, in the form:

$$U(r, \nu) = \frac{fm_0m}{r} \sum_{k=0}^{\infty} A_{2k} X_{2k}(\nu) \frac{R^k}{r^k}.$$

But if $\ell \leq R$, we will call the body "compact" or "flat", and the corresponding expansion of the potential, on the assumption that $c = R$, will be in the form

$$U(r, \nu) = \frac{fm_0m}{r} \sum_{k=0}^{\infty} A_k X_k(\nu) \frac{R^k}{r^k}$$

and, correspondingly, for a body possessing symmetry relative to the center of inertia, in the form

$$U(r, \nu) = \frac{fm_0m}{r} \sum_{k=0}^{\infty} A_{2k} X_{2k}(\nu) \frac{R^k}{r^k}.$$

It stands to reason that in the formulas for the coefficients of A_k it is also necessary to assume $c = 1$ for "lengthened" and $c = R$ for "compact" bodies.

The expansions obtained for the mutual potential of the sphere and body of rotation do not, of course, depend on the selection of a system of coordinates, which can therefore be taken arbitrarily, and the choice of which is determined only by the requirements and convenience in the given problem. For the expansions which we have in mind it is convenient to take a system of coordinates with invariable directions of the axes and with a beginning either in the center of the sphere or in some other point in the space.

Pausing at the latter assumption we will designate by ξ_0 , η_0 and ζ_0 the coordinates of the center of the sphere, and by ξ , η and ζ the coordinates of point G. Then

$$r^2 = (\xi - \xi_0)^2 + (\eta - \eta_0)^2 + (\zeta - \zeta_0)^2,$$

$$\nu = \cos \alpha = \frac{\xi - \xi_0}{r} a_{13} + \frac{\eta - \eta_0}{r} a_{23} + \frac{\zeta - \zeta_0}{r} a_{33}.$$

where a_{13} , a_{23} and a_{33} are the directing cosines of the axis of rotation of the body in the same system of coordinates. If ψ and ϑ are the ordinary angles of precession and nutation of the body, then

$$a_{13} = \sin \psi \sin \vartheta, \quad a_{23} = -\cos \psi \sin \vartheta, \quad a_{33} = \cos \vartheta.$$

In conclusion we give the expansion of the potential in two separate cases where the body is maximally lengthened and where it is maximally flat.

To illustrate the first case we give the expansion of the potential of a homogeneous straight-line segment with a length of 2 and mass m . This expansion, for a case where the point G is the middle of the segment, has the form [2];

$$U = \frac{fm_0m}{r} \sum_{k=0}^{\infty} \frac{X_{2k}(\nu)}{2k+1} \frac{r^{2k}}{r^{2k}}.$$

To illustrate the second case we give the expansion of the potential of a material, homogeneous, flat circular disc of radius R and mass m . This expansion, for a case where the point G is the center of the disc, is written in the form:

$$U = \frac{fm_0m}{r} \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{X_{2k}(\nu)}{k+1} \frac{R^{2k}}{r^{2k}}.$$

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